Higher Order Derivatives

The SECOND DERIVATIVE of a function is the derivative of its first derivative. The second derivative of a function, $f''(x)$, gives the rate of change of the rate of change, $f'(x)$ of the original function *f* (*x*) .

Notation for the second derivative of the function y = f(x):

$$f''(x) = \frac{\partial^2 y}{\partial x^2} = y'' = D_x^2 y$$

$f(x)$ = the position of an object

$f'(x)$ = the velocity of an object (the rate of change of the position of the object)

$f''(x)$ = acceleration of the object (the rate of change of the velocity)

examples:

1. find $f''(x)$ for the function $f(x) = 2x^5 - 3x^2 + 6x$

2. Find $\frac{\partial^2 y}{dx^2}$ for $y = \sqrt{x^2 + 3}$

3. find $D_x^2\left(\frac{1}{x} + \sqrt[3]{x}\right)$

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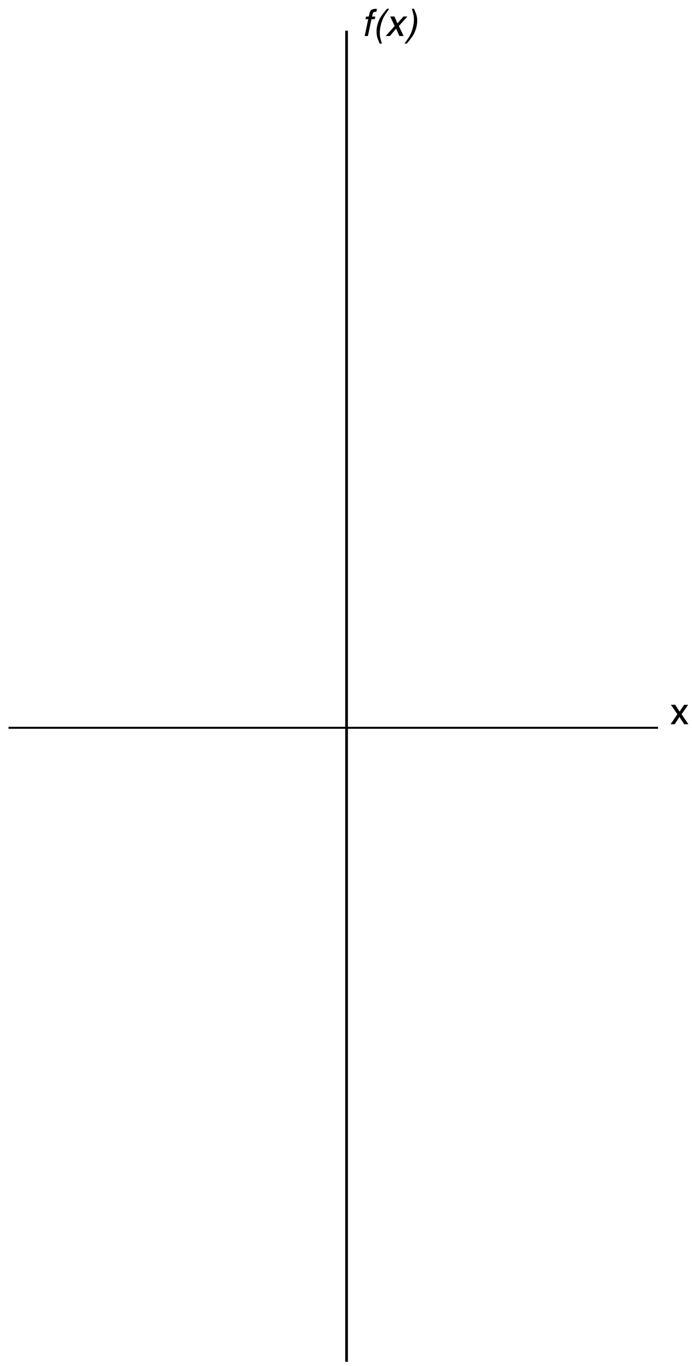
The n^{th} derivative is the resulting function obtained by differentiating n times $\frac{\partial^n}{\partial x^n}$ or $f^n(x)$

CONCAVITY AND INFLECTION POINTS

EXAMPLE: $f(x) = \frac{1}{3}x^3 + x + 2$

find $f'(x)$ = ______________



Observations:

A) for *x* in the interval $(-\infty, 0)$

1.

2.

3.

B) for x in the interval $(0, \infty)$

1.

2.

3.

Concavity and Inflection points

To find where a function is concave upward and concave downward and find inflection points of a function
:

1. Find the domain of $f(x)$
2. Find the **second** derivative of $f(x)$
3. Find the partition numbers for $f''(x)$
   values of x that make
   $f''(x) = 0$
   or
   $f''(x) =$ ND
4. Make a sign analysis for $f''(x)$
5. Intervals where $f''(x) > 0$ are intervals where $f(x)$
   is **concave upward**
   Intervals where $f''(x) < 0$ are intervals where $f(x)$ i
   is **concave downward**
6. Identify any **hypercritical** values
   partition numbers from $f''(x)$ that are in the domain of $f(x)$.
7. Hypercritical points where the sign of $f''(x)$ changes from positive to negative or changes from negative to positive are called **inflection points**.
   Hypercritical values are the <u>only</u> <u>candidates</u> for inflection points

examples: Determine the intervals where $f(x)$ is concave up, concave down, and any inflection points

1. $f(x) = x^4 - 4x^3 - 18x^2 + 1$

2. $f(x) = 5 + x - 3x^2$

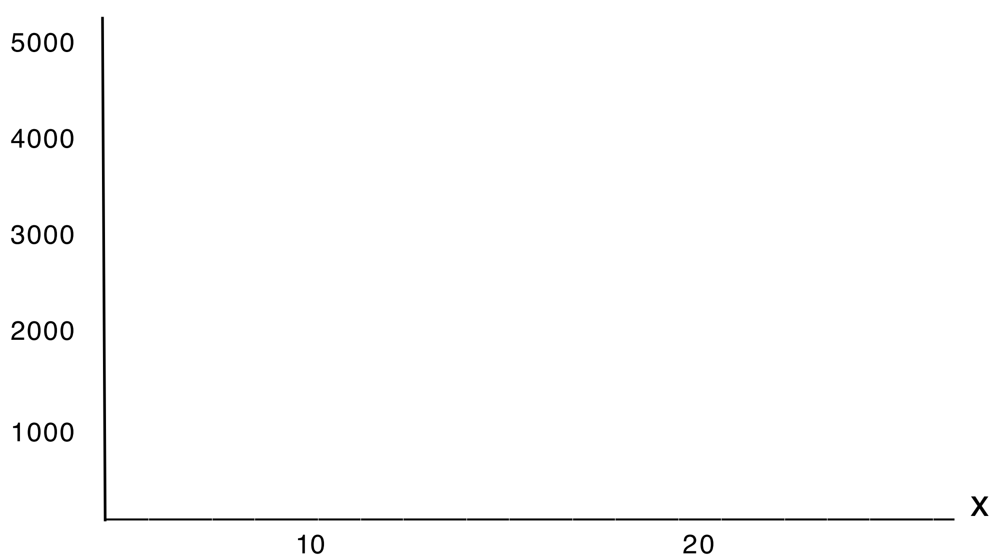
3. $y = \left(x^2 + 5\right)^2$

4. A company estimates that it will sell $N(x)$ units of a product after spending $N(x) = 1000 + 30x^2 - x^3$. for $0 \le x \le 20$

a. When is the rate of change of sales $(N'(x))$ increasing?
decreasing?

b. Find the inflection point(s) for the graph of $N(x)$

c. Sketch the graphs of N and N' on the axes provided below



d. Define the maximum rate of change of sales. Where does this occur?

Second Derivative Test for Local Extrema

Suppose that f is a function, f' and f'' exist on the interval (a,b), c is in (a,b), and $f'(c) = 0$:

1. If $f''(c) > 0$ then $f(c)$ is a local minimum $\underset{c}{\cup}$

2. If $f''(c) < 0$ then $f(c)$ is a local maximum $\underset{c}{\cap}$

3. If $f''(c) = 0$ then the test fails to give any information about local extrema

example: Use the Second Derivative Test to find all local extrema, if the test applies. Otherwise , use the First Derivative Test

1. $f(x) = \frac{2}{3}x^3 - x^2 - 4x - 2$

2. $f(x) = 3x + \frac{12}{x}$

3. $f(x) = \frac{2}{3}x^3 + x^4$

Curve Sketching (4.2)

Find the domain of f

Find the partition numbers for f'

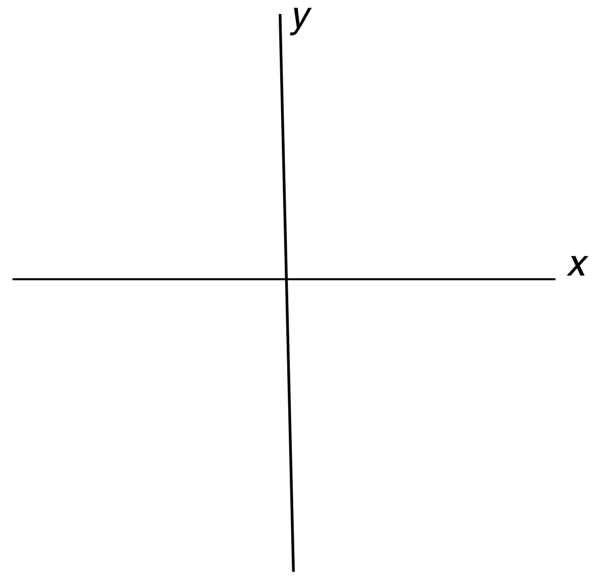
Make a sign analysis for f' and determine where f is increasing, decreasing, and any local extrema

Find the partition numbers for f''

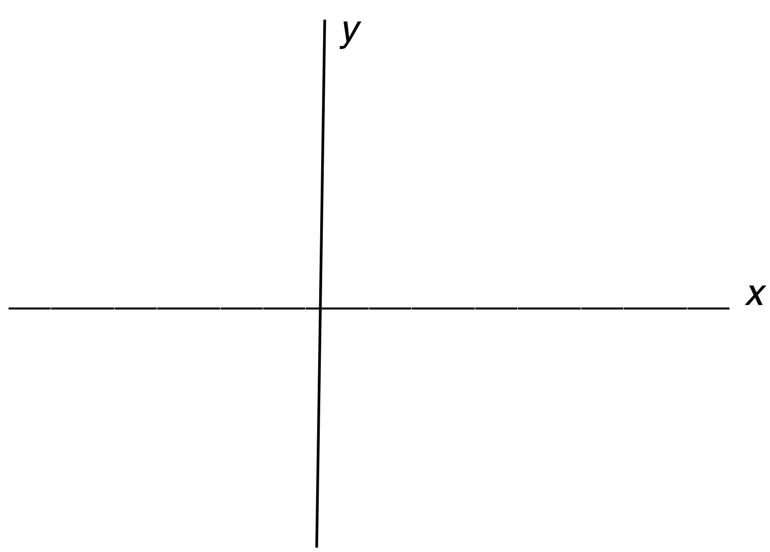
Make a sign analysis for f'' and determine where f is concave upward, concave downward, and any inflection points

Graph each function

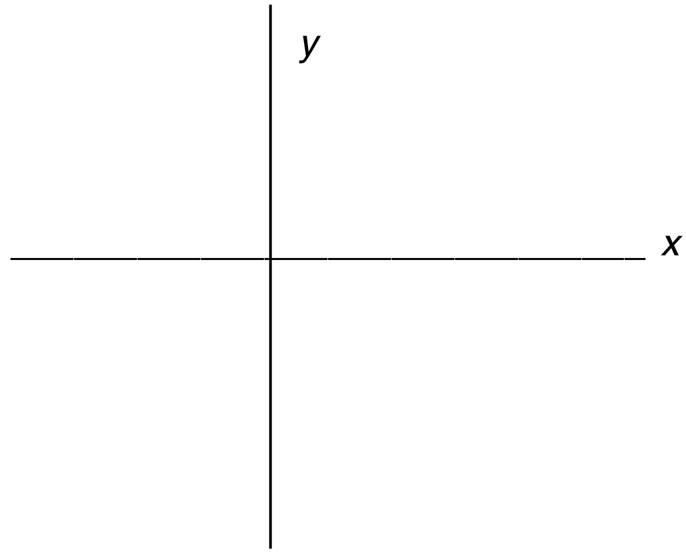
1. $y = 2x^3 + 9x^2 - 3x + 2$



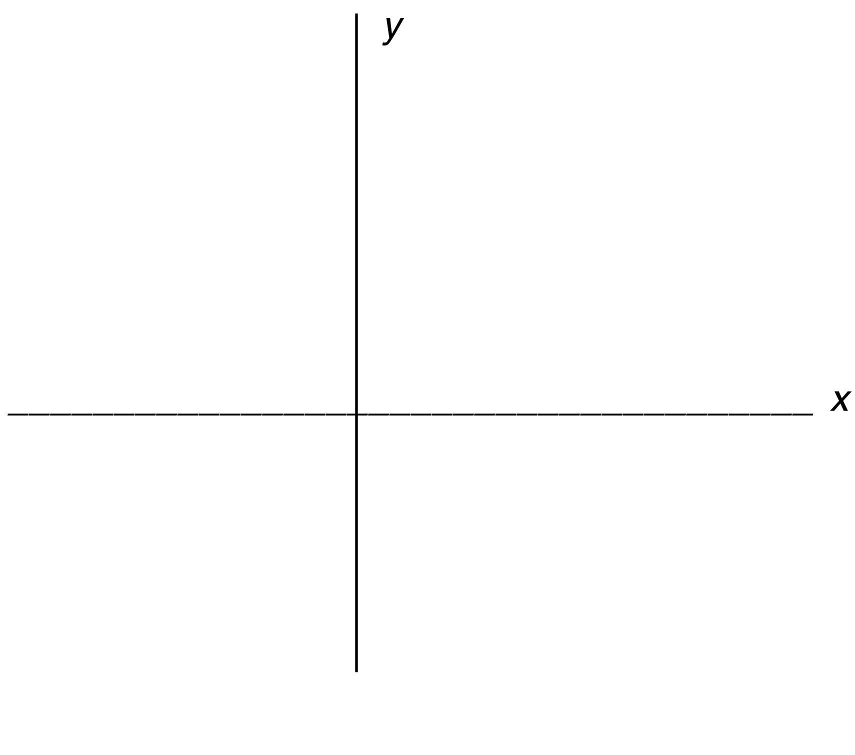
2. $y = 1 - 3x - x^3$



3. $f(x) = \left(x^2 - 9\right)\left(x^2 + 1\right)$



4. $f(x) = -x^4 - 4x^3$



Rational Functions

1. The population of a bacterial colony increases at an increases rate for 1 hour, after which it continues to increase but at a rate that gradually decreases toward zero. Sketch a possible graph for the population *P(t)* as a function of time t .

2. A company determines that t months after production begins on a new product, the number of units produced will be P thousand per month, where $P(t) = \frac{t}{(t+1)^2}$.

a. Find $P'(t)$ and make a sign analysis for P'

b. Find $P''(t)$ and make a sign analysis for P''

c. Sketch a graph of P(t)

d. What happens to production in the long run (ie: as $t \to \infty$)

3. After t hours on a treadmill, a patient's cholesterol level is N(t) gram, where $N(t) = \dfrac{2 + 0.4t}{t + 1}$ for $t \geq 0$

a. Find $N'(t)$ and make a sign analysis for N'

b. Find $N''(t)$ and make a sign analysis for N''

c. Graph N(t).

4. The total cos of producing x units per month is given by $C(x) = 4000 + 10x + 0.1x^2$. Find the minimum average cost.

BUSINESS APPLICATIONS

1. A farmer estimates that if he plants 30 grapefruit trees per acre, the average yield per tree will be 480 pounds. For each additional tree planted per acre, the yield will be reduced by 12 pounds. How many trees should be planted per acre to maximize the yield?

2. A 300 room hotel in Las Vegas is filled to capacity every night at $80 per room. For each $1 increase in rent, 3 fewer rooms are rented. If each rented room costs $10 a day to service

a. how much should the management charge for each room to maximize the profit?

b. how much is the maximum profit?