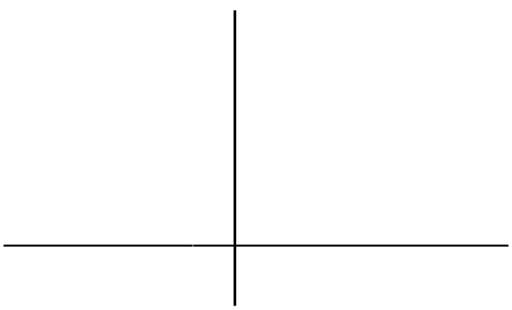
A function of the form $f(x) = c \cdot b^{rx}$, where *b* is a positive real number $(b \neq 1)$, and *c* and *r* are real constants, is called an **exponential function with base *b***.

Recall: <u>polynomial functions</u> such as $y = x^3$ or $y = x^4$, etc. are such that the base is a variable and the exponent is a constant.

Now, with <u>exponential functions</u>, we reverse the roles of the base and the exponent and we have the form $y = b^x$ where the base is a constant and the exponent is a variable.

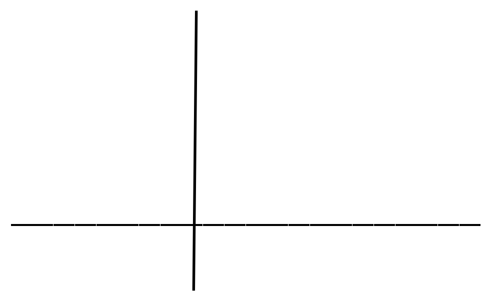
Graph each exponential function on your calculator and copy it below

$y = 2^x$



note: the base is b > 1

$y = \left(\frac{1}{2}\right)^x = 2^{-x}$



note: the base is 0 < b < 1

Properties of the Exponential Function $f(x) = c \cdot b^{rx}$:

domain:___________ range: _______________

y intercept: __________ horizontal asymptote: ___________

for b > 1 : when c=1 and r=1, the slope is positive and y is increasing and concave upward

for 0 < b < 1: when c=1 and r=1, the slope is negative and y is decreasing and concave downward

Compound Interest Formula: For principal P invested at annual rate r (in decimal form) compounded n times per year for t years the amount A is given by $A = P\left(1 + \frac{r}{n}\right)^{n \cdot t}$

Example: What is the amount in an account after 6.5 years if $8000 is invested an an annual rate of 5.5% when

a. compounded monthly?

b. compounded daily?

Continuous Compound Interest Formula:
For principal P invested at annual rate r (in decimal form) compounded continuously for t years, the amount is $A = Pe^{r \cdot t}$.

To see how the number **e** occurs in an application, begin with the formula for compound interest $P\left(1 + \frac{r}{n}\right)^{n \cdot t}$. Suppose that annual interest of 100% so that r = 1 . Suppose also that you can deposit only $1 at this rate, and for only one year. Thus r = 1, P = 1, and t = 1. So, our compounding formula becomes $1\left(1 + \frac{1}{n}\right)^{1 \cdot n} = \left(1 + \frac{1}{n}\right)^{n}$

Use your graphing calculator and enter $\left(1+\frac{1}{n}\right)^n$ into y_1. Evaluate y_1 and complete the chart (to 5 or 6 decimal places

n	$\left(1+\frac{1}{n}\right)^n$
1	
10	
100	
200	
500	
1,000	
$n \to \infty$	

The table suggests that as n gets larger and larger, then $\left(1+\frac{1}{n}\right)^n$ becomes closer to a number whose approximate value is 2.718281828.......This number is referred to as **e**. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

Graph each

$y = e^x$ $y = e^{-x}$

Example: What is the amount in an account after 6.5 years if $8000 is invested an an annual rate of 5.5% if the money is compounded continuously? (compare results to that on page 2 of Ch5 notes)

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The formula $A = Pe^{r \cdot t}$ to calculate future value of an account that will result from an investment of P (present value)

Solve the equation $A = Pe^{r \cdot t}$ for P:

$$A = Pe^{r \cdot t}$$
$$\frac{A}{e^{rt}} = \frac{Pe^{rt}}{e^{rt}}$$
$$P = \frac{A}{e^{rt}} = Ae^{-rt}$$

ex. How much money should be invested today at 6.5%, compounded continually, so that 20 years from now it will be worth $20,000?

ex. The value *V* of a machine at the end of *t* years is given by $V = C(1-r)^t$ where *C* is the original cost of the machine and *r* is the rate of depreciation. Find the value at the end of 4 years of a machine that costs $6000 originally

ex.(refer to the previous ex.) A machine 8 years old is valued at $3000. If the rate of depreciation is 15%, find the original cost of the machine.

Logarithmic functions

The **exponential form** $b^y = x$ is equivalent to the
logarithmic form $y = \log_b x$ (b is a positive constant other that 1)

Exponential Equation	Logarithmic Equation
$5^2 = 25$	$\log_5 25 = 2$
$2^{-3} = \frac{1}{8}$	______________
$9^{\frac{1}{2}} = 3$	______________
________________	$\log_{10} 100 = 2$
________________	$\log_4 8 = \frac{3}{2}$
________________	$\log_e 148.4131591... = 5$

>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

NOTE: $\log_{10} x$ can be written as $\log x$
and
$\log_e x$ can be written as $\ln x$

In calculus, the most frequently used logarithmic base is e . In this case $\log_e x$ or $\ln x$ is called the **natural logarithm** of x

ex. Write $e^2 = 7.389056099...$ in its logarithmic form ______________

ex. Write $e^0 = 1$ in its logarithmic form _____________

ex. Write $e^{-1} = 0.3678794412...$ in its logarithmic form _____________

ex. Write $e^1 = e$ in its logarithmic form _________________

The algebra of the NATURAL LOGARITHM Function

The domain of $f(x)=\ln x$ is the set of all positive real numbers. For positive real numbers M and N:

Property	**Example**
1. $\ln(MN)=\ln M+\ln N$	________________________
2. $\ln\left(\frac{M}{N}\right)=\ln M-\ln N$	________________________
3. $\ln\left(M^r\right)=r\cdot\ln M$	________________________
4. $\ln M=\ln N$ if and only if M=N	________________________

5. $\ln 1=0$ 6. $\ln e=1$ 7. $e^{\ln x}=x$ 8. $\ln e^x=x$

Examples: Solve for x:

1. $e^{x+2}=127$

2. $2=1.06^x$

3. $6e^{-0.4x}=48$

4. $10^{1-2x}=72$

Examples:

1. A deposit of $1000 is made into an account that pays interest at a rate of 4.5% compounded continuously. How long will it take for the balance to be $7500?

2. How long will it take money to double if it is invested at 6% interest compounded continuously?

3. If $3000 is to be invested and compounded continuously, what rate will be necessary for the money to increase to $8000 in 12 years?

Differentiation of Logarithmic Functions

If $f(x) = \ln x$ then $f'(x) = \frac{1}{x}$

examples:

1. find $\frac{\partial y}{\partial x}$ for $y = 4x^2 + \ln x$

2. . find $\frac{\partial y}{\partial x}$ for $y = x^3 \cdot \ln x$

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If $y = \ln(g(x))$ then $y' = \frac{1}{g(x)} \cdot g'(x)$

examples

1. . find $\frac{\partial y}{\partial x}$ for $y = \ln x^3$

2. . find $\frac{\partial y}{\partial x}$ for $y = \ln\left(3x^2 - 4x\right)$

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Combining Rules:

ex . find $\frac{\partial y}{\partial x}$ for $y = x^2 \cdot \ln x$

Examples: Find the absolute extrema for each of the functions on the indicated intervals

1. $f(x) = x^2 - 8\ln x$ [0.3 , 4]

2. $f(x) = \dfrac{\ln x}{x^2}$ [1 , 2]

>>

Use the first and second derivatives to sketch the graph of $f(x) = x^2 \cdot \ln x$

Differentiation of Exponential Functions

If $f(x) = e^x$ then $f'(x) = e^x$

and

If $f(x) = e^{g(x)}$ then $f'(x) = e^{g(x)} \cdot g'(x)$

examples Find the derivative of each function

1. $f(x) = -5e^x$

2. $f(x) = 8e^{2x}$

3. $f(x) = 6e^{x^2}$

4. $f(x) = e^{3x+4x^2}$

ex 5. Determine the slope of the graph of the function $f(x) = e^x \cdot \ln(x+4)$ at x = 2.

ex. 6. Determine the interval(x) where the function $f(x) = e^{-0.4x^2}$ is increasing and decreasing and any local extrema

A function $y = f(t)$ satisfies the equation $\frac{\partial y}{\partial t} = ky$ if and only if $y = ce^{kt}$. If *k* and *c* are positive we say that **y grows exponentially**. (or increases exponentially)

ex. Biologists have determined that when sufficient space and nutrients are available, the number of bacteria in a culture grows exponentially. Suppose that 2,000 bacteria are initially present in a certain culture and that 6,000 are present 20 minutes later.

a. Find an exponential function that represents the population at time t where t is measured in minutes

b. How many bacteria will be present at the end of 1 hour?

>>

A function $y = f(t)$ satisfies the equation $\frac{\partial y}{\partial t} = -ky$ if and only if $y = ce^{-kt}$. If *k* and *c* are positive we say that **y decays exponentially**. (or declines exponentially)

ex. A radioactive substance decays exponentially. If 500 grams of the substance were present initially and 400 grams are present 50 years later

a. Find an exponential function that represents the population at time t where t is measured in years

b. how many grams will be present after 200 years?