FUNCTIONS OF SEVERAL VARIABLES

Evaluate each function at the indicated point

1. $z = f(x,y) = x^2 + xy + y^3$ a. f(2,1) = ____ b. f(1,2)= ______

2. $F(t,w) = \frac{4w}{2-t^2}$ a. f(3, -1)=______

3. $f(x,y) = xe^y + \ln x$ a. $f\left(e^2, \ln 2\right) =$ ______

EX 1. A population that grows exponentially satisfies $Q(Q_0,k,t) = Q_0 e^{kt}$ (Q_0 is the initial amount, *k* is the relative (per capita) growth rate and *t* is time in years form now. The population of a certain county is currently 5 million people and is growing at the rate of 3% per year. What will the population be in 7 years?

P (____, ____ , _____) = _______

EX 2. The Cobb-Douglas Production function states that *x* units of labor and *y* units of capital will produce *f(x,y)* items, where $f(x,y) = kx^n y^{1-n}$, where *k* and *n* are constants. If $f(x,y) = 5x^{0.7}y^{0.3}$, how many items are produced using 8 units of labor and 10 units of capital?

EX 3. A company produces and sells *x* pounds of caramel at \$*p* per pound and *y* pounds of chocolate at \$*q* per pound according to the demand functions:
$$p = 20 - 3x + y$$
$$q = 15 + x - 2y$$
.

If the total cost function is $C(x,y) = 17 + 5x + 4y$

a. find the Revenue function R(x,y) b. find the Profit function P(x,y)

PARTIAL DERIVATIVES

1. For the function $f(x,y) = 9 - 5x + 3y^2$, find

   a. $f_x(x,y)=$ b. $f_y(x,y)=$

   c. $f_x(3,2) =$ _____ d. $f_y(3,2) =$ _____

2. For the function $f(x,y) = xy$, find

   a. $f_x(x,y)=$ b. $f_y(x,y)=$

3. For the function $f(x,y) = 3xy$ find

   a. $f_x(x,y)=$ b. $f_y(x,y)=$

4. For the function $f(x,y) = x^2y$ find

   a. $f_x(x,y)=$ b. $f_y(x,y)=$

5. For the function $f(x,y) = x^3y^5$ find

   a. $f_x(x,y)=$ b. $f_y(x,y)=$

6. For the function $f(x,y) = 6x^3y^4$ find

   a. $f_x(x,y)=$ b. $f_y(x,y)=$

7. For the function $f(x,y) = x^3 + 2xy - y^2 + x - 4y + 3$, find

   a. $\frac{\partial f}{\partial x} =$ b. $\frac{\partial f}{\partial y} =$

   c. $f_x(4,1) =$ _____ d. $f_y(3,1) =$ _____

8 For the function $f(x, y) = \sqrt{9 - x^2 - y^2}$, find

a. $\dfrac{\partial f}{\partial x} =$

b. $\dfrac{\partial f}{\partial y} =$

c. $f_x(1,2) =$ _____

d. $f_y(1,2) =$ _____

9. If $z = f(x,y) = xe^{-2xy}$ find

a. $f_x(x,y)=$

b. $f_y(x,y)=$

10. If $z = f(x,y) = \ln(5x - 4y)$ find

a. $f_x(x,y)=$

b. $f_y(x,y)=$

11. For the function $f(x,y) = e^{3x+6y}$ find

a. $f_x(x,y)=$

b. $f_y(x,y)=$

EX 12. For $C(x,y) = 2x^2 + 2xy + 3y^2 - 16x - 18y + 54$, find the values of *x* and *y* for which <u>both</u> $C_x(x,y) = 0$ and $C_y(x,y) = 0$

EX 13. A company produces x standard items selling at $*p* each and *y* luxury items selling at $*q* each according to the demand functions

$$p = 100 - 3x + y$$

$$q = 80 + x - 2y$$

with total cost function $C(x,y) = 400 + 8x + 10y$

a. Find the Revenue function R(x,y)

b. Find $R_x(5,10)$ and interpret

c. Find $R_y(5,10)$ and interpret

d. Find the Profit function P(x,y)

e. Find $P_x(5,10)$ and interpret

f. Find $P_y(5,10)$ and interpret

EX 14. A ski manufacturer has weekly production function $S = f(x,y) = 250x^{0.63}y^{0.37}$ where S is the number of pairs of skis produced using x units of labor and y units of capital

a. Find the **marginal productivity of labor** $\frac{\partial S}{\partial x} =$

b. Find the **marginal productivity of capital** $\frac{\partial S}{\partial y} =$

c. find $f_x(10,20) =$

d. find $f_y(10,20) =$

e. To increase productivity faster, should labor or capital be increased?

SECOND ORDER PARTIAL DERIVATIVES

If z = f(x,y) then

$$f_{xx} = f_{xx}(x,y) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$f_{yy} = f_{yy}(x,y) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Mixed Partial Derivatives

$$f_{xy} = f_{xy}(x,y) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

first differentiate with respect to x, holding y constant. Then differentiate with respect to y, holding x constant

$$f_{yx} = f_{yx}(x,y) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

first differentiate with respect to y, holding x constant. Then differentiate with respect to x, holding y constant

---

Find all second order partial derivatives for each function

1. $f(x,y) = y^4 - 2xy^2 + 5x^2 + 15x - 8y$

$f_x(x,y) =$ $f_y(x,y) =$

$f_{xx}(x,y) =$ $f_{yy}(x,y) =$

$f_{xy}(x,y) =$ $f_{yx}(x,y) =$

2. $f(x,y) = 2ye^{xy} + x^2$

$f_x(x,y) =$ $f_y(x,y) =$

$f_{xx}(x,y) =$ $f_{yy}(x,y) =$

$f_{xy}(x,y) =$ $f_{yx}(x,y) =$

Algebra techniques for systems of equations:

If each equation contains only one variable, factor and solve it.
If both equations are linear, use the addition (or elimination) method
If one or both equations are nonlinear, use substitution from the 'easier' one into the 'harder' one.

SOLVE EACH SYSTEM OF EQUATIONS

1. $6x^2 - 18x + 12 = 0$
   $2y - 10 = 0$

2. $2x + y - 4 = 0$
   $x + 2y - 5 = 0$

3. $6x^2 - 6y + 6 = 0$
   $-6x + 6y - 18 = 0$

4. $-100x + 100y + 20 = 0$
   $100x - 140y + 80 = 0$

LOCAL EXTREMA FOR FUNCTIONS OF TWO VARIABLES

SECOND PARTIALS TEST (or D-test)

STEPS:

1. find $f_x(x,y)$ and $f_y(x,y)$ (first partials)
2. set both first partials equal to 0 and solve to find all critical points (a,b)
3. find the second partials: f_{xx}, f_{yy}, f_{xy}
4. substitute each critical value into the chart to find D:

$$D = f_{xx}(a,b) \cdot f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

CASE 1. If $D > 0$ and $f_{xx}(a,b) > 0$, then f(a , b) is a **local minimum**

CASE 2. If $D > 0$ and $f_{xx}(a,b) < 0$, then f(a , b) is a **local maximum**

CASE 3. If $D < 0$ then f(a , b) is a **saddle point**

CASE 4. If $D = 0$ then this test gives no information

>>

EX.1 Classify the critical points of $f(x,y) = 2x^3 + y^2 - 9x^2 - 10y + 12x - 2$

C.P.	$f_{xx} =$	$f_{yy} =$	$f_{xy} =$	D	conclusion

EX.2 Classify the critical points of $f(x,y) = x^2 + xy + y^2 - 4x - 5y$

C.P.	$f_{xx} =$	$f_{yy} =$	$f_{xy} =$	D	conclusion

EX.3 Classify the critical points of $f(x,y) = 2x^3 - 6xy + 3y^2 + 6x - 18y$

C.P.	$f_{xx} =$	$f_{yy} =$	$f_{xy} =$	D	conclusion

Applications of the Second Partials Test

Ex1. A company manufactures x ten-speed bicycles sell at \$*p* and y three-speed bicycles selling at \$*q*. The weekly demand and costs equations are

$p = 230 - 9x + y$

$q = 130 + x - 4y$

$C(x,y) = 200 + 80x + 30y$

a. How many of each type of bicycle should be sold to maximize the weekly profit?

b. What is the weekly maximum profit?

>>

Ex.2 The owner of a business advertises in the newspaper and on the radio. He has found that the number of units that he sells is modeled by $N(x,y) = -0.5x^2 - y^2 + 8x + 12y + 240$, where x (in thousands of dollars) is the amount spent on newspaper ads and y (in thousands of dollars) is the amount spent on radio ads.

a. How much should he spend on each to maximize the number of units sold?

b. What will the maximum number of units sold?

Lagrange Multipliers

In the previous section, we optimized functions of several variables. However, in many practical optimization problems, we must maximize or minimize a function in which the independent variables are subjected to certain further **constraints**. These extremum of a function *f* is called a **constrained relative extrema of *f***. We will use a method called the **Method of Lagrange Multipliers**, named for Joseph Lagrange, a French mathematician.

EX. The Acrosonic Company manufactures a bookshelf loudspeaker system that may be bought full assembled or as a kit. The total weekly profit (in dollars) that the company realized in producing and selling the systems is given by the function

$P(x,y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 120x + 100y - 5000$ where x denotes the number of full assembled units and y denotes the number of kits produced and sold per week.

The company's management has decided that the production of these systems should be restricted to a total of 230 per week. Under this constraint, how many assembled units and how many kits should be produced per week to maximize their weekly profit?

The problem is equivalent to maximizing the function P(x,y) , subject to the constraint x+y = 230

To find this constrained relative maximum of P(x,y), we use the **Method of Lagrange Multipliers**:

step 1: Form the Lagrange function: $F(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y)$, where the Greek letter lambda, λ, is called the **Lagrange Multiplier** and $g(x,y) = 0$ is the **constraint** .

step 2: Find each of the first partials: F_x , F_y , and F_λ

step 3: Solve the system for the three equations:

$$F_x(x,y,\lambda) = 0$$
$$F_y(x,y,\lambda) = 0$$
$$F_\lambda(x,y,\lambda) = 0$$

step 4: If f hs a local maximum or local minimum subject to the constraint g(x,y)=0, the corresponding x and y values will be found among the solutions to the system in step 3.

Example (continued)

In this example: for the constraint g(x,y)=0, we use $g(x,y) = x + y - 230$

$F(x,y,\lambda) =$

solve the system of the three equations:

$F_x(x,y,\lambda) =$

$F_y(x,y,\lambda) =$

$F_\lambda(x,y,\lambda) =$

(first, isolate λ in the equations F_x and F_y and then substitute into the equation F_λ)

The required constrained relative maximum of P occurs at the critical point (180, 50). Thus the company's profit is maximized by producing 180 assembled units and 50 kits. *NOTE: We can use the D-test to verify this*

The maximum weekly profit is P(180, 50) = $10,312.50

The absolute value of the Lagrange multiplier, λ , is called the **marginal productivity of money**. Thus in this example $|\lambda|$ = the number of additional units of the objective function, *P*, for each additional unit of the constraint function, *g(x,y)* In this example $|\lambda|$ *= ?*

Example: The total monthly profit of Robertson Controls Company in manufacturing and selling x hundred of its standard mechanical setback thermostats and y hundred of its deluxe electronic setback thermostats per month is given by the total profit function

$P(x,y) = -\frac{1}{8}x^2 - \frac{1}{2}y^2 - \frac{1}{4}xy + 13x + 40y - 280$, where P is in hundreds of dollars.

If the production of setback thermostats is to restricted to a total of exactly 40 per month

a. how many of each model should the company manufacture in order to maximize its monthly profits?
b. what is the maximum monthly profit?
c. find the value of the marginal productivity of money and explain.